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Generalised P-representations in quantum optics

P D Drummond† and C W Gardiner
Department of Physics, University of Waikato, Hamilton, New Zealand

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Abstract. A class of normal ordering representations of quantum operators is introduced, that generalises the Glauber–Sudarshan P-representation by using nondiagonal coherent state projection operators. These are shown to have practical application to the solution of quantum mechanical master equations. Different representations have different domains of integration, on a complex extension of the usual canonical phase-space. The 'complex P-representation' is the case in which analytic P-functions are defined and normalised on contours in the complex plane. In this case, exact steady-state solutions can often be obtained, even when this is not possible using the Glauber–Sudarshan P-representation. The 'positive P-representation' is the case in which the domain is the whole complex phase-space. In this case the P-function may always be chosen positive, and any Fokker–Planck equation arising can be chosen to have a positive-semidefinite diffusion array. Thus the 'positive P-representation' is a genuine probability distribution. The new representations are especially useful in cases of nonclassical statistics.

1. Introduction

The coherent state basis has proved invaluable in quantum optics and quantum statistical mechanics (Glauber 1963a, b, Sudarshan 1963).

Using this basis, phase-space Fokker–Planck equations can be developed that correspond to quantum master equations for the density operator (Haken 1970, Louisell 1974). The Fokker–Planck equations are relatively simple, and observables can be directly calculated as correlations of the distribution function, or P-function.

In general, however, the Glauber–Sudarshan P-representation—which is a diagonal expansion of the density operator in coherent states—results in a distribution that can have negative values and delta-function singularities. This occurs especially in cases of nonclassical photon statistics, recently observed in experiments (Kimble *et al* 1978, Leuchs *et al* 1979) on atomic fluorescence as predicted by Carmichael and Walls (1976). The purpose of the present paper is to introduce a class of generalised P-representations, which are well-behaved even when the Glauber–Sudarshan P-representation is singular. The new P-functions have similar time-development equations and observables to those of the Glauber–Sudarshan P-function. However, they are defined in a complex phase-space instead of a real (classical) phase-space. This allows the solution of quantum optical problems involving nonclassical photon statistics, in a straightforward way.

We believe the introduction of a reasonably rigorous basis for these methods is timely, since they have been in use without proof for some two years already (Chaturvedi *et al* 1977, Drummond and Carmichael 1978, Drummond *et al* 1979, Drummond

† Present address: University of Rochester, Department of Physics and Astronomy, Rochester, New York 14627, USA.

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and Walls 1980). This paper will not therefore give any detailed examples, since the usefulness of the methods has already been amply demonstrated in these earlier works.

An example of the problems that arise in using the Glauber–Sudarshan P-representation is the steady state of a coherently driven single mode interferometer with a nonlinear absorber. The Fokker–Planck equation† that results was obtained by Chaturvedi *et al* (1977):

$$\frac{\partial}{\partial t}P(\boldsymbol{\alpha})=\left\{-\frac{\partial}{\partial\alpha}[E-K\alpha-2\chi\alpha^2\alpha^*]-\frac{\partial^2}{\partial\alpha^2}[\chi\alpha^2]+\mathrm{cc}\right\}P(\boldsymbol{\alpha}). \tag{1.1}$$

Here E is the driving amplitude, and K, χ are the linear and nonlinear rates of absorption respectively. The corresponding equation in real variables is obtained by transforming to the classical phase-space of position and momentum:

$$x=(\alpha+\alpha^*)/\sqrt{2}, \qquad p=(\alpha-\alpha^*)/\mathrm{i}\sqrt{2} \tag{1.2}$$

$$\begin{aligned}\frac{\partial}{\partial t}P(x,p)=\bigg\{&-\frac{\partial}{\partial x}[E-Kx-\chi x(p^2+x^2)]-\frac{\chi}{2}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial p^2}\right)(x^2-p^2)\\&-\frac{\partial}{\partial p}[-Kp-\chi p(p^2+x^2)]-2\chi\frac{\partial}{\partial x}\frac{\partial}{\partial p}(px)\bigg\}P(x,p).\end{aligned} \tag{1.3}$$

This real variable equation clearly has a non-positive-definite diffusion term.

For this reason, there is no corresponding stochastic differential equation on the real phase-space, and in fact no smooth normalisable steady-state distribution exists.

If the lack of positive-definiteness was ignored, a naïve application of stochastic theory would yield an Itô stochastic differential equation (Arnold 1974), with complex noise terms. In the (α, α^*) variables, this would have the following form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha\\ \alpha^*\end{pmatrix}=\begin{pmatrix}E-K\alpha-2\chi\alpha^2\alpha^*\\ E-K\alpha-2\chi\alpha^{*2}\alpha\end{pmatrix}+\begin{pmatrix}\mathrm{i}(2\chi)^{1/2}\alpha\xi_1(t)\\ \mathrm{i}(2\chi)^{1/2}\alpha^*\xi_2(t)\end{pmatrix}. \tag{1.4}$$

Here (ξ_1, ξ_2) are independent Gaussian stochastic functions‡, whose correlations are defined by:

$$\langle\xi_i(t)\xi_j(t')\rangle=\delta_{ij}\delta(t-t'). \tag{1.5}$$

However this naïve procedure would cause a paradox to arise: the equations no longer allow α and α^* to remain complex conjugate because $\xi_1(t)$, $\xi_2(t)$ are independent. It can be noted that this procedure has been used in laser theory (Louisell 1974, Haken 1970). In the case of laser theory the non-positive-definite terms are normally negligible, which is not the case in the present example.

We will show later that in fact equation (1.4) does give formally correct results, provided (α, α^*) are replaced by (α, β), which are independent complex variables. This equation will be obtained in a rigorous way by using a generalised P-representation that includes both diagonal and nondiagonal terms in the coherent-state expansion of the density operator.

† Throughout this paper, differential operators in Fokker–Planck equations act on all terms in products, including $P(\boldsymbol{\alpha})$.

‡ The correlations in this case are different from those of a classical noise source, in which (ξ_1, ξ_2) would normally be correlated.

An alternative approach would be to regard (α, α^*) as independent variables in equation (1.1). This means that an independent variable (β) would be substituted for (α^*), and moments would be defined as:

$$\langle(\hat{a}^+)^n(\hat{a})^m\rangle = \iint_{C,C'} (\beta)^n(\alpha)^m P(\alpha, \beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta. \tag{1.6}$$

We will show later that this is also a correct procedure which can be derived by a nondiagonal expansion of the density operator in coherent states. In fact this procedure leads to an exact solution for the steady-state moments, in the case of equation (1.1) when C, C' are defined as independent (non complex-conjugate) analytic contours.

2. Generalised *P*-representations

The quantum statistics of a single mode of the electromagnetic field (as well as other quantum problems involving bosons) is equivalent to that of an harmonic oscillator with annihilation and creation operators $(\hat{a}, \hat{a}^+)$. All physical observables are obtained from the multinomial moments and correlations of $(\hat{a}, \hat{a}^+)$ (Glauber 1963a). These in turn are determined using the quantum density operator $\hat{\rho}$, and it is often simplest to represent $\hat{\rho}$ using a distribution function over a c-number phase-space. It is usual to expand $\hat{\rho}$ with the aid of the coherent states, defined as eigenstates of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.1}$$

The Glauber–Sudarshan *P*-representation is an expansion in diagonal coherent state projection operators:

$$\hat{\rho} = \int \mathrm{d}^2\alpha\; |\alpha\rangle\langle\alpha| P(\alpha, \alpha^*). \tag{2.2}$$

Because of the overcompleteness of the coherent states, the diagonal *P*-function $P(\alpha, \alpha^*)$ is not unique, and does not always exist as a well-behaved function (although Klauder and Sudarshan (1970) have shown that it does exist in terms of distributions with singularities). Glauber (1963b) recognised these problems, and introduced the R-representation:

$$\hat{\rho} = \frac{1}{\pi^2}\iint \mathrm{d}^2\alpha\; \mathrm{d}^2\beta\, R(\alpha^*, \beta) \exp[-(|\alpha|^2 + |\beta|^2)/2]|\alpha\rangle\langle\beta| \tag{2.3}$$

which he showed always to exist, and to be unique provided that $R(\alpha^*, \beta)$ is analytic in α^* and β. In spite of this, the R-representation has not been widely used, as Fokker–Planck equations for $R(\alpha^*, \beta)$ do not normally exist.

We shall introduce a class of generalised *P*-representations, related to the R-representation, by expanding in nondiagonal coherent state projection operators. For simplicity, the term '*P*-function' will still be used.

Define:

$$\hat{\Lambda}(\alpha, \beta) = |\alpha\rangle\langle\beta^*|/(\langle\beta^*|\alpha\rangle) \tag{2.4}$$

$$= \exp(\alpha\hat{a}^+ - \alpha\beta)|0\rangle\langle 0| \exp(\beta\hat{a}) \tag{2.5}$$

and

$$\hat{\rho} = \int P(\alpha, \beta)\hat{\Lambda}(\alpha, \beta)\, d\mu\, (\alpha, \beta). \tag{2.6}$$

Here $d\mu$ is an integration measure, and $P(\alpha, \beta)$ is analogous to the usual P-function. The projection operator $\hat{\Lambda}(\alpha, \beta)$ is analytic in (α, β), which will be of significance later.

The integration measure is left undefined at present: by using various integration measures, a class of generalised P-representations is generated, some of which will be investigated in detail in the remainder of this section. Existence theorems for these are obtained in § 3.

2.1. *The Glauber–Sudarshan P-representation*

Let

$$d\mu\,(\alpha, \beta) = \delta^2(\alpha^* - \beta)\, d^2\alpha\, d^2\beta. \tag{2.7}$$

This measure corresponds to the diagonal (Glauber–Sudarshan) P-representation. The existence properties of the corresponding P-function are well known (Glauber 1970, Cahill and Glauber 1969, Klauder and Sudarshan 1970). The representation defined using (2.7) is identical to that of (2.1).

2.2. *Complex P-representation*

$$d\mu\,(\alpha, \beta) = d\alpha\, d\beta. \tag{2.8}$$

Here (α, β) are treated as complex variables which are to be integrated on individual contours C, C'. Theorems 1 and 2 show the existence of this representation under certain circumstances. In particular, the existence of this representation for an operator expanded in a finite basis of number states is of interest: this is a characteristic situation involving possible photon antibunching (anticorrelated photons), where the diagonal Glauber–Sudarshan P-representation would be singular.

It is appropriate to call this representation the complex P-representation (as complex values of $P(\alpha, \beta)$ occur). The representation gives rise to a $P(\alpha, \beta)$ which can be shown to satisfy a Fokker–Planck equation obtained by replacing (α, α^*) with (α, β) in the usual Glauber–Sudarshan type of Fokker–Planck equation.

Under certain circumstances, exact solutions to Fokker–Planck equations occur which cannot be normalised as Glauber–Sudarshan diagonal P-functions. These can be handled with the present representation by choosing appropriate C,C' (paths of integration) in the complex phase-space of (α, β).

2.3. *Positive P-representation*

$$d\mu\,(\alpha, \beta) = d^2\alpha\, d^2\beta. \tag{2.9}$$

This representation allows (α, β) to vary independently over the whole complex plane. Theorems 3 and 4 show that $P(\alpha, \beta)$ always exists for a physical density operator, and can always be chosen positive, in which case we call it the positive P-representation. This means that $P(\alpha, \beta)$ has all the properties of a genuine probability.

It will also be shown that, provided any Fokker–Planck equation exists for time-development in the Glauber–Sudarshan representation, a corresponding Fokker–Planck equation exists with a positive semi-definite diffusion coefficient for the positive P-representation.

The resulting positive semi-definite Fokker–Planck equation will be shown to correspond exactly to the stochastic differential equation given by a naïve application of stochastic theory (for example, to equation (1.1)), with the replacement of (α, α^*) by (α, β).

2.4. *Operator identities*

From the definitions (2.5) of the nondiagonal coherent state projection operators, the following identities can be obtained. For simplicity, $\boldsymbol{\alpha}$ is used to denote (α, β):

$$\begin{aligned}
&\hat{a}\hat{\Lambda}(\boldsymbol{\alpha}) = \alpha\hat{\Lambda}(\boldsymbol{\alpha}) \\
&\hat{a}^+\hat{\Lambda}(\boldsymbol{\alpha}) = (\beta + \partial/\partial\alpha)\hat{\Lambda}(\boldsymbol{\alpha}) \\
&\hat{\Lambda}(\boldsymbol{\alpha})\hat{a}^+ = \hat{\Lambda}(\boldsymbol{\alpha})\beta \\
&\hat{\Lambda}(\boldsymbol{\alpha})\hat{a} = (\partial/\partial\beta + \alpha)\hat{\Lambda}(\boldsymbol{\alpha}).
\end{aligned} \tag{2.10}$$

By substituting the above identities into equation (2.6) defining the generalised P-representation, and using partial integration (provided the boundary terms vanish), these identities can be used to generate operations on the P-function depending on the representation.

2.4.1. *Glauber–Sudarshan P-representation*

$$\begin{aligned}
\hat{a}\hat{\rho} &= \int (\alpha P(\boldsymbol{\alpha}))\hat{\Lambda}(\boldsymbol{\alpha})\, \mathrm{d}\mu(\boldsymbol{\alpha}) \\
\hat{a}^+\hat{\rho} &= \int [(\alpha^* - \partial/\partial\alpha)P(\boldsymbol{\alpha})]\hat{\Lambda}(\boldsymbol{\alpha})\, \mathrm{d}\mu(\boldsymbol{\alpha}) \\
\hat{\rho}\hat{a}^+ &= \int P(\boldsymbol{\alpha})\,.\,\alpha^*\hat{\Lambda}(\boldsymbol{\alpha})\, \mathrm{d}\mu(\boldsymbol{\alpha}) \\
\hat{\rho}\hat{a} &= \int [(\alpha - \partial/\partial\alpha^*)P(\boldsymbol{\alpha})]\hat{\Lambda}(\boldsymbol{\alpha})\, \mathrm{d}\mu(\boldsymbol{\alpha}).
\end{aligned} \tag{2.11}$$

(From now on, only the equivalent operation on $P(\boldsymbol{\alpha})$ will be written explicitly.)

2.4.2. *Complex P-representation*

$$\begin{aligned}
&\hat{a}\hat{\rho} \leftrightarrow \alpha P(\boldsymbol{\alpha}) \\
&\hat{a}^+\hat{\rho} \leftrightarrow (\beta - \partial/\partial\alpha)P(\boldsymbol{\alpha}) \\
&\hat{\rho}\hat{a}^+ \leftrightarrow P(\boldsymbol{\alpha})\beta \\
&\hat{\rho}\hat{a} \leftrightarrow (-\partial/\partial\beta + \alpha)P(\boldsymbol{\alpha}).
\end{aligned} \tag{2.12}$$

2.4.3. *Positive-P representation.* We now use the analyticity of $\hat{\Lambda}(\alpha, \beta)$ and note that if

$$\begin{aligned}
\alpha &= \alpha_x + \mathrm{i}\alpha_y \\
\beta &= \beta_x + \mathrm{i}\beta_y
\end{aligned} \tag{2.13}$$

then

$$(\partial/\partial\alpha)\hat{\Lambda}(\boldsymbol{\alpha}) = (\partial/\partial\alpha_x)\hat{\Lambda}(\boldsymbol{\alpha}) = (-\mathrm{i}\partial/\partial\alpha_y)\hat{\Lambda}/\boldsymbol{\alpha}) \tag{2.14}$$

and

$$(\partial/\partial\beta)\hat{\Lambda}(\boldsymbol{\alpha}) = (\partial/\partial\beta_x)\hat{\Lambda}(\boldsymbol{\alpha}) = (-\mathrm{i}\partial/\partial\beta_y)\hat{\Lambda}(\boldsymbol{\alpha}) \tag{2.15}$$

so that as well as all of (2.12) being true in this case, we also have

$$\begin{aligned} \hat{a}^+\hat{\rho} &\leftrightarrow (\beta - \partial/\partial\alpha_x)P(\boldsymbol{\alpha}) \leftrightarrow (\beta + \mathrm{i}\partial/\partial\alpha_y)P(\boldsymbol{\alpha}) \\ \hat{\rho}\hat{a} &\leftrightarrow (-\partial/\partial\beta_x + \alpha)P(\boldsymbol{\alpha}) \leftrightarrow (\mathrm{i}\partial/\partial\beta_y + \alpha)P(\boldsymbol{\alpha}). \end{aligned} \tag{2.16}$$

The correspondences (2.11) are well known for the Glauber–Sudarshan P-representation, and can be used to obtain equations of motion and observables given the equation of motion (master equation) for $\hat{\rho}$. The new correspondences (2.12, 2.16) can also be used in a similar way to obtain equations of motion for the new representations. Observable properties, as regards moments of the annihilation and creation operators, are given by:

$$\langle(\hat{a}^+)^n\hat{a}^m\rangle = \int P(\boldsymbol{\alpha})\beta^n\alpha^m \,\mathrm{d}\mu(\boldsymbol{\alpha}). \tag{2.17}$$

Equation (2.17) is true for any representation, and also gives the normalisation property:

$$1 = \int P(\boldsymbol{\alpha})\,\mathrm{d}\mu(\boldsymbol{\alpha}). \tag{2.18}$$

3. Existence theorems

The existence properties of the diagonal (Glauber–Sudarshan) P-representation are well known: in many cases this P-function would only exist as a series of derivatives of delta-functions, or as the limit of a sequence. We will now show that the generalised P-representations defined using nondiagonal coherent state projection operators have much stronger existence properties. That is, a generalised P-representation exists, with a smooth P-function, even when the Glauber–Sudarshan P-function would be singular. The existence theorems are as follows:

Theorem 1. A complex P-representation exists for an operator with an expansion in a finite number of number states.

Proof. Let

$$\hat{\rho} = \sum_{n,m}\sum C_{nm}(\hat{a}^+)^m|0\rangle\langle 0|\hat{a}^n. \tag{3.1}$$

Then, by Cauchy's theorem,

$$\hat{\rho} = \oint\oint_{C,C'} \hat{\Lambda}(\boldsymbol{\alpha})P(\boldsymbol{\alpha})\,\mathrm{d}\mu(\boldsymbol{\alpha}) \tag{3.2}$$

with

$$P(\boldsymbol{\alpha}) = (-1/4\pi^2)\,\mathrm{e}^{\alpha\beta} \sum_{n,m}\sum C_{nm}\,.\,n!\,m!\,\alpha^{-m-1}\beta^{-n-1} \tag{3.3}$$

where C, C' are integration paths enclosing the origin. (The corresponding Glauber–Sudarshan P-function is highly singular in this case, as mentioned by Klauder and Sudarshan (1970)).

Theorem 2. A complex P-representation exists for any operator with an expansion on a bounded range of coherent states, i.e. for

$$\hat{\rho} = \iint_{D,D'} \hat{\Lambda}(\alpha, \beta) C(\alpha, \beta)\, d^2\alpha\, d^2\beta \tag{3.4}$$

where D, D' are bounded in each complex plane.

Proof. Application of Cauchy's theorem shows that if

$$P(\boldsymbol{\alpha}) = -\frac{1}{4\pi^2} \iint_{D,D'} [C(\alpha', \beta')/(\alpha - \alpha')(\beta - \beta')]\, d^2\alpha'\, d^2\beta' \tag{3.5}$$

then

$$\hat{\rho} = \iint_{C,C'} \hat{\Lambda}(\boldsymbol{\alpha}) P(\boldsymbol{\alpha})\, d\alpha\, d\beta \tag{3.6}$$

where C, C' enclose D, D' respectively. Hence the complex P-representation exists in this case relative to any bounded expansion in coherent state projection operators.

Theorem 3. Whenever a Glauber–Sudarshan P-representation exists, a corresponding positive (nondiagonal) P-representation exists, with $P(\boldsymbol{\alpha})$ given by:

$$P(\boldsymbol{\alpha}) = (1/4\pi^2) \exp(-|\alpha - \beta^*|^2/4) \langle \tfrac{1}{2}(\alpha + \beta^*)|\hat{\rho}|(\alpha + \beta^*)/2\rangle. \tag{3.7}$$

Proof. $P(\boldsymbol{\alpha})$ is real and positive by definition, since $\hat{\rho}$ is an Hermitian, positive-definite operator. Let $P'(\boldsymbol{\alpha})$ be the Glauber–Sudarshan P-function. Then by direct substitution into (3.7):

$$P(\boldsymbol{\alpha}) = [1/4\pi^2] \int P'(\alpha', \alpha'^*) \exp(-|\alpha - \alpha'|^2/2 - |\beta^* - \alpha'|^2/2)\, d^2\alpha'. \tag{3.8}$$

It is next necessary to demonstrate that $P(\boldsymbol{\alpha})$ as defined does represent $\hat{\rho}$, so the RHS of equation (2.6) is evaluated:

$$\iint \hat{\Lambda}(\boldsymbol{\alpha}) P(\boldsymbol{\alpha})\, d\mu(\boldsymbol{\alpha})$$

$$= \iiint \frac{d^2\alpha'\, d\mu(\boldsymbol{\alpha})}{4\pi^2} \hat{\Lambda}(\boldsymbol{\alpha}) P'(\alpha', \alpha'^*) \exp[-|\alpha - \alpha'|^2/2 - |\beta^* - \alpha'|^2/2]. \tag{3.9}$$

Now the projector $\hat{\Lambda}(\boldsymbol{\alpha})$ is analytic in (α, β) so the following identity can be used (for any analytic function $f(\alpha)$):

$$f(\alpha') = \frac{1}{2\pi} \int f(\alpha) \exp(-|\alpha - \alpha'|^2/2)\, d^2\alpha'. \tag{3.10}$$

Hence

$$\iint \hat{\Lambda}(\boldsymbol{\alpha}) P(\boldsymbol{\alpha})\, d^2\alpha\, d^2\beta = \int |\alpha'\rangle\langle\alpha'| P'(\alpha', \alpha'^*)\, d^2\alpha' = \hat{\rho}. \tag{3.11}$$

This demonstrates that equation (3.7) does provide a real, positive representation of $\hat{\rho}$; provided a Glauber–Sudarshan P-representation exists.

Theorem 4. A positive P-representation exists for any quantum density operator.

Proof. Define $P(\boldsymbol{\alpha})$ using equation (3.7) in Theorem 3. In order to show that this represents a quantum density operator in the general case, the characteristic function:

$$\chi(\lambda)\equiv \mathrm{Tr}(\hat{\rho}\,\mathrm{e}^{\lambda\hat{a}^+}\,\mathrm{e}^{-\lambda^*\hat{a}}) \tag{3.12}$$

is used. This has been shown by Glauber (1970) to define the density operator uniquely. In terms of the R-representation for $\hat{\rho}$, the characteristic function is:

$$\chi(\lambda)=\int R(\alpha^*, \lambda+\alpha)\exp(-\lambda^*\alpha-|\alpha|^2)(\mathrm{d}^2\alpha/\pi). \tag{3.13}$$

We now substitute the R-representation for $\hat{\rho}$ into equation (3.7), which defines $P(\boldsymbol{\alpha})$ in terms of the diagonal matrix elements of $\hat{\rho}$. We then define $\hat{\rho}_P$ to be given by the positive P-representation form (2.6), calculate the corresponding characteristic function $\chi_P(\boldsymbol{\lambda})$ using (3.12) and show this is the same as the original characteristic function for $\hat{\rho}$. Thus:

$$\chi_P(\lambda)\equiv\iint P(\boldsymbol{\alpha})\exp(\lambda\beta-\lambda^*\alpha)\,\mathrm{d}^2\alpha\,\mathrm{d}^2\beta \tag{3.14}$$

$$=\left(\frac{1}{4\pi^4}\right)\iiiint R(\alpha'^*,\beta')\exp\{\lambda\beta-\lambda^*\alpha-|\alpha|^2/2-|\beta|^2/2-|\alpha'|^2-|\beta'|^2$$
$$+\beta'^*(\alpha+\beta^*)/2+\alpha'^*(\alpha^*+\beta)/2\}\,\mathrm{d}^2\alpha\,\mathrm{d}^2\beta\,\mathrm{d}^2\alpha'\,\mathrm{d}^2\beta'. \tag{3.15}$$

We now make a variable change by defining:

$$\begin{aligned}&\gamma=(\alpha+\beta^*)/2 \qquad \delta=(\alpha-\beta^*)/2\\ &\therefore \alpha=(\gamma+\delta) \qquad \beta^*=(\gamma-\delta)\\ &\mathrm{d}^2\alpha\,\mathrm{d}^2\beta=4\,\mathrm{d}^2\gamma\,\mathrm{d}^2\delta.\end{aligned} \tag{3.16}$$

Noting that R is an analytic function, the following identity is useful

$$R(\alpha^*,\gamma)=\frac{1}{\pi}\int R(\alpha^*,\beta)\exp(\gamma\beta^*-|\beta|^2)\,\mathrm{d}^2\beta. \tag{3.17}$$

Hence the above expression for the characteristic function can be simplified to give

$$\chi_P(\lambda)=\frac{1}{\pi^3}\iiint R(\alpha'^*,\gamma)\exp[\lambda(\gamma-\delta)^*-\lambda^*(\gamma+\delta)$$
$$-|\gamma|^2-|\delta|^2-|\alpha'|^2+\alpha'\gamma^*]\,\mathrm{d}^2\gamma\,\mathrm{d}^2\delta\,\mathrm{d}^2\alpha' \tag{3.18}$$

$$=\frac{1}{\pi^2}\iint R(\alpha'^*,\gamma)\exp(|\lambda|^2+\lambda\gamma^*-\lambda^*\gamma-|\gamma|^2$$
$$-|\alpha'|^2+\alpha'\gamma^*)\,\mathrm{d}^2\gamma\,\mathrm{d}^2\alpha' \tag{3.19}$$

$$=\frac{1}{\pi}\int R(\alpha^*,\lambda+\alpha)\exp(-\lambda^*\alpha-|\alpha|^2)\,\mathrm{d}^2\alpha \tag{3.20}$$

$$\therefore\quad \chi_P(\lambda)=\mathrm{Tr}(\hat{\rho}\,\mathrm{e}^{\lambda\hat{a}^+}\,\mathrm{e}^{-\lambda^*\hat{a}})=\chi(\lambda). \tag{3.21}$$

The last step follows from the identity for the characteristic function defined relative to the Glauber R-representation, as given previously.

This result is more general than Theorem 3, as it demonstrates that a positive P-representation always exists with a smooth positive P-function regardless of the existence properties of the Glauber–Sudarshan P-representation.

Theorem 5. The operator $\hat{\rho}$ can be expanded directly in terms of its diagonal coherent state matrix elements.

Proof. This is a trivial corollary of Theorem 4. From equations (3.7) and (2.6)

$$\hat{\rho} = \frac{1}{\pi^2} \iint \left(\frac{|\gamma+\delta\rangle\langle\gamma-\delta|}{\langle\gamma-\delta|\gamma+\delta\rangle} \right) \mathrm{e}^{-|\delta|^2} \langle\gamma|\hat{\rho}|\gamma\rangle \, \mathrm{d}^2\gamma \, \mathrm{d}^2\delta. \tag{3.22}$$

The above theorem in fact states a well known fact in quantum optics, namely that the diagonal coherent state matrix elements of $\hat{\rho}$ define $\hat{\rho}$ completely. A similar equation (although not identical) was obtained by Lonke (1978) using special function techniques. The present proof of this theorem, however, provides an elegant application of a generalised P-representation.

4. Time development equations

Using the operator correspondences appropriate to the Glauber–Sudarshan P-representation, a whole formalism has been developed (Lax 1968, Haken 1970, Louisell 1973) for converting quantum mechanical master equations into Fokker–Planck equations, which, as noted earlier, do not always have positive definite diffusion coefficients. Other representations on classical phase spaces were developed by Cahill and Glauber (1969), and more generally, by Agarwal and Wolf 1970.† These representations also allow phase-space time development equations which may be of Fokker–Planck form, with diffusion coefficients that are not necessarily positive definite. We shall now consider what the corresponding equations are in the case of the generalised P-representations.

4.1. *Complex-P representation*

Here the procedure yields a very similar equation to that for the diagonal case. We assume that, by appropriate re-ordering of the differential operators, we can reduce the quantum mechanical master equation to the form (where $(\alpha, \beta) = \boldsymbol{\alpha} \equiv (\alpha^{(1)}, \alpha^{(2)})$; $\mu = 1, 2$):

$$\frac{\partial \hat{\rho}}{\partial t} = \iint_{C,C'} \hat{\Lambda}(\boldsymbol{\alpha}) \frac{\partial P(\boldsymbol{\alpha})}{\partial t} \, \mathrm{d}\alpha \, \mathrm{d}\beta$$

$$= \iint_{C,C'} \left\{ \left(A^{\mu}(\boldsymbol{\alpha}) \frac{\partial}{\partial \alpha^{\mu}} + \tfrac{1}{2} D^{\mu\nu}(\boldsymbol{\alpha}) \frac{\partial}{\partial \alpha^{\mu}} \frac{\partial}{\partial \alpha^{\nu}} \right) \hat{\Lambda}(\boldsymbol{\alpha}) \right\} P(\boldsymbol{\alpha}) \, \mathrm{d}\alpha \, \mathrm{d}\beta. \tag{4.1}$$

† Classical phase-space representations of this type may be calculated from the characteristic function provided the relevant Fourier integral exists.

We now integrate by parts, and, if we can neglect boundary terms, which may be made possible by an appropriate choice of contours C, C', at least one solution is obtained by equating the coefficients of $\hat{\Lambda}(\boldsymbol{\alpha})$

$$\frac{\partial P(\boldsymbol{\alpha})}{\partial t}=\left[-\frac{\partial}{\partial\alpha^{\mu}}A^{\mu}(\boldsymbol{\alpha})+\frac{1}{2}\frac{\partial}{\partial\alpha^{\mu}}\frac{\partial}{\partial\alpha^{\nu}}D^{\mu\nu}(\boldsymbol{\alpha})\right]P(\boldsymbol{\alpha}). \tag{4.2}$$

This equation is sufficient to imply equation (4.1), but is not a unique equation because the $\hat{\Lambda}(\boldsymbol{\alpha})$ are not linearly independent. It should be noted that for this complex P-representation, $A^{\mu}(\boldsymbol{\alpha})$ and $D^{\mu\nu}(\boldsymbol{\alpha})$ are always analytic in $\boldsymbol{\alpha}$; hence if $P(\boldsymbol{\alpha})$ is initially analytic, (4.2) preserves this analyticity as time develops.

Applying this procedure to equation (1.1) would yield the following equation:

$$\frac{\partial P(\boldsymbol{\alpha})}{\partial t}=\left[\frac{\partial}{\partial\alpha}(K\alpha+2\chi\alpha^2\beta-E)-\chi\left(\frac{\partial^2}{\partial\alpha^2}\alpha^2+\frac{\partial^2}{\partial\beta^2}\beta^2\right)+\frac{\partial}{\partial\beta}(K\beta+2\chi\beta^2\alpha-E)\right]P(\boldsymbol{\alpha}). \tag{4.3}$$

This has the following exact steady state solution:

$$P(\boldsymbol{\alpha})=(\alpha)^{\mu-2}(\beta)^{\mu^*-2}\exp[(E/\chi)(1/\alpha+1/\beta)+2\alpha\beta] \tag{4.4}$$

where $\mu=(K/\chi)$.

It can be seen immediately that this potential would diverge if the Glauber–Sudarshan representation was used, with $\beta\equiv\alpha^*$. Instead it is necessary to choose alternative paths of integration for (α,β) that are to be line integrals on the individual (α,β) complex planes. This is straightforward, with the result that appropriate integration paths are Hankel paths (Abramowitz and Stegun 1964) in the variables $(1/\alpha, 1/\beta)$ (Drummond 1979). A calculation of the correlation functions is given in the Appendix. The physical interpretation of these results will be treated in subsequent work.

4.2. *Positive P-representation*

We assume that the same equation (4.1) is being considered. The symmetric matrix can always be factorised into the form

$$\mathbf{D}(\boldsymbol{\alpha})=\mathbf{B}(\boldsymbol{\alpha})\mathbf{B}^{\mathrm{T}}(\boldsymbol{\alpha}). \tag{4.5}$$

We now write

$$\boldsymbol{A}(\boldsymbol{\alpha})=\boldsymbol{A}_x(\boldsymbol{\alpha})+\mathrm{i}\boldsymbol{A}_y(\boldsymbol{\alpha}) \tag{4.6}$$

$$\mathbf{B}(\boldsymbol{\alpha})=\mathbf{B}_x(\boldsymbol{\alpha})+\mathrm{i}\mathbf{B}_y(\boldsymbol{\alpha}) \tag{4.7}$$

where $\boldsymbol{A}_x$, $\boldsymbol{A}_y$, $\mathbf{B}_x$, $\mathbf{B}_y$ are real. We then find that the master equation (4.1) yields

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t}&=\iint \mathrm{d}^2\alpha\ \mathrm{d}^2\beta\ \hat{\Lambda}(\boldsymbol{\alpha})(\partial P(\boldsymbol{\alpha})/\partial t)\\ &=\iint P(\boldsymbol{\alpha})\{A_x^{\mu}(\boldsymbol{\alpha})\partial_{\mu}^{x}+A_y^{\mu}(\boldsymbol{\alpha})\partial_{\mu}^{y}+\tfrac{1}{2}\{B_x^{\mu\sigma}B_x^{\nu\sigma}\partial_{\mu}^{x}\partial_{\nu}^{x}\\ &\quad+B_y^{\mu\sigma}B_y^{\nu\sigma}\partial_{\mu}^{y}\partial_{\nu}^{y}+2B_x^{\mu\sigma}B_y^{\nu\sigma}\partial_{\mu}^{x}\partial_{\nu}^{y}\}\}\hat{\Lambda}(\boldsymbol{\alpha})\ \mathrm{d}^2\alpha\ \mathrm{d}^2\beta.\end{aligned} \tag{4.8}$$

Here we have, for notational simplicity, written $\partial/\partial\alpha_x^{\mu}=\partial_{\mu}^{x}$ etc, and have used the analyticity of $\hat{\Lambda}(\boldsymbol{\alpha})$ to make either of the replacements

$$\partial/\partial\alpha^{\mu}\leftrightarrow\partial_{\mu}^{x}\leftrightarrow-\mathrm{i}\partial_{\mu}^{y} \tag{4.9}$$

in such a way as to yield (4.8). Now, provided partial integration is permissible, we deduce the Fokker–Planck equation:

$$\partial P(\boldsymbol{\alpha})/\partial t = [-\partial_\mu^x A_x^\mu(\boldsymbol{\alpha}) - \partial_\mu^y A_y^\mu(\boldsymbol{\alpha}) + \tfrac{1}{2}\{\partial_\mu^x \partial_\nu^x B_x^{\mu\sigma}(\boldsymbol{\alpha}) B_x^{\nu\sigma}(\boldsymbol{\alpha}) + 2\partial_\mu^x \partial_\nu^y B_x^{\mu\sigma}(\boldsymbol{\alpha}) B_y^{\nu\sigma}(\boldsymbol{\alpha}) + \partial_\mu^y \partial_\nu^y B_y^{\mu\sigma}(\boldsymbol{\alpha}) B_y^{\nu\sigma}(\boldsymbol{\alpha})\}]P(\boldsymbol{\alpha}). \tag{4.10}$$

Again, this is not a unique time-development equation but (4.8) is a consequence of (4.10).

However, the Fokker–Planck equation (4.10) now possesses a positive semidefinite diffusion matrix in a four-dimensional space whose vectors are

$$(\alpha_x^{(1)}, \alpha_x^{(2)}, \alpha_y^{(1)}, \alpha_y^{(2)}) \equiv (\alpha_x, \beta_x, \alpha_y, \beta_y). \tag{4.11}$$

We find the drift vector is:

$$\mathscr{A}(\boldsymbol{\alpha}) \equiv (A_x^{(1)}(\boldsymbol{\alpha}), A_x^{(2)}(\boldsymbol{\alpha}), A_y^{(1)}(\boldsymbol{\alpha}), A_y^{(2)}(\boldsymbol{\alpha})) \tag{4.12}$$

and the diffusion matrix is:

$$\mathscr{D}(\boldsymbol{\alpha}) \equiv \begin{bmatrix} \mathbf{B}_x \,.\, \mathbf{B}_x^{\mathrm{T}}, & \mathbf{B}_x \,.\, \mathbf{B}_y^{\mathrm{T}} \\ \mathbf{B}_y \,.\, \mathbf{B}_x^{\mathrm{T}}, & \mathbf{B}_y \,.\, \mathbf{B}_y^{\mathrm{T}} \end{bmatrix}(\boldsymbol{\alpha}) \equiv \mathscr{B}(\boldsymbol{\alpha})\mathscr{B}^{\mathrm{T}}(\boldsymbol{\alpha}) \tag{4.13}$$

where:

$$\mathscr{B}(\boldsymbol{\alpha}) \equiv \begin{bmatrix} \mathbf{B}_x, & 0 \\ \mathbf{B}_y, & 0 \end{bmatrix}(\boldsymbol{\alpha}) \tag{4.14}$$

and $\mathscr{D}$ is thus explicitly positive semidefinite (and not positive definite). The corresponding Itô stochastic differential equations can be written:

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{\alpha}_x \\ \boldsymbol{\alpha}_y \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_x(\boldsymbol{\alpha}) \\ \boldsymbol{A}_y(\boldsymbol{\alpha}) \end{pmatrix} + \begin{pmatrix} \mathbf{B}_x(\boldsymbol{\alpha}) \,.\, \boldsymbol{\xi}(t) \\ \mathbf{B}_y(\boldsymbol{\alpha}) \,.\, \boldsymbol{\xi}(t) \end{pmatrix} \tag{4.15}$$

or, recombining real and imaginary parts,

$$\partial \boldsymbol{\alpha}/\partial t = \boldsymbol{A}(\boldsymbol{\alpha}) + \mathbf{B}(\boldsymbol{\alpha}) \,.\, \boldsymbol{\xi}(t). \tag{4.16}$$

Apart from the substitution $\alpha^* \to \beta$, equation (4.16) is just the stochastic differential equation which would be obtained by using the Glauber–Sudarshan representation, and naively converting the Fokker–Planck equation with a non-positive-definite diffusion matrix into an Itô stochastic differential equation.

In our derivation, the two formal variables (α, α^*) have been replaced by variables in the complex plane, (α, β) that are allowed to fluctuate independently. The positive P-representation as defined here thus appears as a mathematical justification of this procedure. In the case of equation (1.4) the replacement would simply result in the following:

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} E - K\alpha - 2\chi\alpha^2\beta \\ E - K\beta - 2\chi\beta^2\alpha \end{pmatrix} + \begin{pmatrix} 2\mathrm{i}\chi\alpha\xi_1(t) \\ 2\mathrm{i}\chi\beta\xi_2(t) \end{pmatrix}. \tag{4.17}$$

This equation has been treated asymptotically (in a linearised approximation valid for large photon number) by Chaturvedi *et al* (1977). These authors demonstrate that the resulting steady state correlation functions have the nonclassical statistical property of photon antibunching.

4.3. Application of the positive P-representation

We have shown:

(a) By Theorems 3 and 4 of § 3, that for any $\hat{\rho}$ there exists a positive P-representation, with a positive normalisable P which is a function of two complex variables.

(b) If a Fokker–Planck equation exists, it can be cast in a form with real coefficients and a positive semidefinite diffusion matrix. Thus if P is initially positive, it stays positive.

We have not shown that the explicit functional form of the P-function given in Theorems 3 and 4 is preserved by these equations, and in fact this is not the case in general. Thus it is clear that there is no unique positive P-function corresponding to a given $\hat{\rho}$. However, the different possible P-functions do give the same observable properties, with regard to the multinomial moments and correlation functions of the annihilation and creation operators. Non physical moments of the form $\langle(\alpha_x)^n\rangle$ or $\langle(\alpha_y)^n\rangle$ are not unique for $n>1$.

It is interesting to compare this situation with that for the Q-representation, as defined as:

$$Q(\alpha)=\langle\alpha|\hat{\rho}|\alpha\rangle. \tag{4.18}$$

While the Q-representation is a unique, positive representation, it does not necessarily have a positive-definite Fokker–Planck equation or a corresponding stochastic differential equation. Furthermore, as this is an antinormally ordered representation, both the time-development equations and the expressions for normally ordered observables are complicated by extra derivative terms that are not present for generalised P-representations as defined here.

It is evident that the lack of uniqueness of generalised P-representations is not a disadvantage. Instead, it allows a freedom of choice in defining time-development equations which is extremely valuable in calculations for practical applications.

Although the present paper is the first publication of the theoretical background of this approach in quantum optics, the technique of defining a complex phase-space with an equivalent positive-definite Fokker–Planck equation and stochastic equation has been used in earlier work.

The usefulness of stochastic processes in the complex plane was first demonstrated without proof, by Gardiner and Chaturvedi (1977), Chaturvedi and Gardiner (1978), in work on the number state master equations of chemical reaction theory. The method was generalised to quantum optical applications by Chaturvedi *et al* (1977), Drummond and Carmichael (1978), Drummond *et al* (1979), Drummond and Walls (1980) in work on the nonlinear absorber, cooperative fluorescence, instabilities in nonlinear optics, and on optical bistability. A review of the application to a nonlinear interferometer is given in the Appendix.

5. Discussion

A class of representations has been introduced, that generalises the Glauber–Sudarshan P-representation. These representations are normal-ordering representations, defined on an integration domain that is a complex phase-space. The phase-space may be regarded as a complex extension of a classical phase-space, in which the canonical position and momentum variables now have complex parts.

Existence theorems have been derived showing that these generalised P-representations have stronger existence properties than the Glauber–Sudarshan P-representation (which is defined on a real phase-space).

That is, non-singular P-functions for generalised P-representations exist even when the Glauber–Sudarshan P-function would be singular. An example of this is the nonlinear absorber (equation 1.1). While the diagonal P-function would be singular in this case, a smooth normalisable distribution can be found using a complex P-representation.

The strongest existence theorem shows that at least one generalised P-function exists in all cases, that is real and positive, and its form is given explicitly. It is also shown that one can always choose a Fokker–Planck equation that is explicitly positive-semidefinite, even though (because of non-uniqueness), this Fokker–Planck equation does not necessarily preserve equation (3.7). The usefulness of positive-definite Fokker–Planck equations is that they allow the use of path-integrals (Graham 1977) and stochastic differential equations (Arnold 1974), which provide asymptotic approximation schemes (Chaturvedi and Gardiner 1978) even when an exact solution for a nonlinear quantum system would be impractical.

In terms of application of the methods outlined here, the main change in deriving and solving Fokker–Planck equations is the replacement of the complex-conjugate pair (α, α^*) with a non complex-conjugate pair (α, β). This may be written as (α, α^+) provided it is recognised that while (α, α^+) represent the Hermitian adjoint pair $(\hat{a}, \hat{a}^+)$ these new variables are not themselves complex conjugates, except in the mean.

Thus Fokker–Planck equations on complex manifolds or stochastic differential equations on a complex phase-space can equally be used, with resulting moments obtained as:

$$\langle(\hat{a}^+)^m(\hat{a})^n\rangle = \int (\alpha^+)^m(\alpha)^n P(\alpha, \alpha^+)\,\mathrm{d}\mu\,(\alpha, \alpha^+). \tag{5.1}$$

The results of this paper are readily generalised to the case of an n-mode quantum system. The generalised P-representation for n-modes is defined analogously to the n-mode R-representation of Glauber (1963b) as follows:

$$\hat{\rho} = \int \hat{\Lambda}(\alpha_1, \alpha_1^+, \alpha_2, \alpha_2^+, \ldots)P(\alpha_1, \alpha_1^+, \ldots \alpha_n^+)\,\mathrm{d}\mu\,(\alpha_1, \alpha_1^+ \ldots \alpha_n^+)$$

where:

$$\hat{\Lambda}(\boldsymbol{\alpha}) \equiv \frac{|\alpha_1, \alpha_2 \ldots \alpha_n\rangle\langle(\alpha_1^+)^*, (\alpha_2^+)^* \ldots (\alpha_n^+)^*|}{\langle(\alpha_1^+)^*, (\alpha_2^+)^* \ldots (\alpha_n^+)^*|\alpha_1, \alpha_2 \ldots \alpha_n\rangle}. \tag{5.2}$$

Here (α_j, α_j^+) are the c-number analogues of the jth pair of annihilation and creation operators $(\hat{a}_j, \hat{a}_j^+)$, and $P(\alpha_1, \alpha_1^+ \ldots \alpha_n^+)$ is a distribution in the space $\mathbb{C}^{2n}$ (complex space of $2n$ dimensions) that represents the n-mode quantum density operator.

In this case all the previous results relating to existence properties and positive-definiteness of the Fokker–Planck equation have a trivial extension. These results apply equally to n-modes, with the replacement of $\boldsymbol{\alpha} = (\alpha, \alpha^+)$ by $\boldsymbol{\alpha} = (\alpha_1, \alpha_1^+, \ldots \alpha_n, \alpha_n^+)$.

In a subsequent paper we will derive techniques, arising from these generalised P-representations, which can be applied to multi-time correlation functions.

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Appendix

The steady-state problem of a coherently driven single-mode interferometer with a nonlinear medium is a good application of the generalised P-representation, as all orders of moments of the steady-state distribution are easily calculated. Here the interferometer is assumed to be a high-Q interferometer, where the medium response is much faster than the decay-rate of the quasi-mode of the cavity being driven. In addition, both the nonlinear dispersion and absorption during a transit-time of the interferometer must be relatively small, in order for the single-mode theory to be valid. The resulting model Hamiltonian including nonlinear polarisability and absorption is:

$$\begin{aligned}
\hat{H} &= \sum_{j=1}^{5} \hat{H}_j \\
\hat{H}_1 &= \hbar\omega_1 \hat{a}^+ a \\
\hat{H}_2 &= \hbar\chi'' \hat{a}^{+2}\hat{a}^2 \\
\hat{H}_3 &= \mathrm{i}\hbar(\mathscr{E}\,\mathrm{e}^{-\mathrm{i}\omega t}\hat{a}^+ - \mathscr{E}^*\,\mathrm{e}^{\mathrm{i}\omega t}\hat{a}) \\
\hat{H}_4 &= \hat{a}^+\hat{\Gamma}_1 + \hat{a}\hat{\Gamma}_1^+ \\
\hat{H}_5 &= \hat{a}^{+2}\hat{\Gamma}_2 + \hat{a}^2\hat{\Gamma}_2.
\end{aligned} \tag{A1}$$

Here ω_1 is the relevant interferometer resonance frequency for a single (transverse and longitudinal) quasi-mode, with the same polarisation as that of the input field. The coefficient $\mathscr{E}$ is related to the coupled power P of the input beam by $|\mathscr{E}|^2 = P(1-R)/(\hbar\omega_1\Delta^+)$, for a round-trip time Δt and mirror reflectivity R. The driving laser is coherent, with frequency ω. The operators $\hat{\Gamma}_1$, $\hat{\Gamma}_2$ are reservoirs for one-photon and two-photon interactions respectively. The parameter χ'' is related to the nonlinear polarisability of the medium. These terms are explained in detail in Chaturvedi *et al* (1977) and Drummond and Walls (1980). Using standard techniques the equation of motion for $\hat{\rho}$ is obtained in the Markovian, rotating wave approximation and the interaction picture:

$$\begin{aligned}
\frac{\partial\hat{\rho}}{\partial t} &= \sum_{j=1}^{5} \hat{\mathscr{L}}_j[\hat{\rho}j] \\
\hat{\mathscr{L}}_1[\hat{\rho}] &= -\mathrm{i}\Delta\omega[\hat{a}^+\hat{a}, \hat{\rho}] \\
\hat{\mathscr{L}}_2[\hat{\rho}] &= -\mathrm{i}\chi''[\hat{a}^{+2}\hat{a}^2, \hat{\rho}] \\
\hat{\mathscr{L}}_3[\hat{\rho}] &= [\mathscr{E}\hat{a}^+ - \mathscr{E}^*\hat{a}, \hat{\rho}] \\
\hat{\mathscr{L}}_4[\hat{\rho}] &= \kappa'(2\hat{a}\hat{\rho}\hat{a}^+ - \hat{\rho}\hat{a}^+\hat{a} - \hat{a}^+\hat{a}\hat{\rho}) \\
\hat{\mathscr{L}}_5[\hat{\rho}] &= \chi'(2\hat{a}^2\hat{\rho}\hat{a}^{+2} - \hat{\rho}\hat{a}^{+2}\hat{a}^2 - \hat{a}^{+2}\hat{a}^2\hat{\rho}).
\end{aligned} \tag{A2}$$

In these equations, $\Delta\omega = \omega_1 - \omega$, and κ', χ' are the relaxation rates due to the reservoirs $\hat{\Gamma}_1$, $\hat{\Gamma}_2$ respectively (the reservoir temperatures T_1, T_2 must satisfy $kT_1 \ll \hbar\omega$,

$kT_2 \ll 2\hbar\omega$). The question is, then, how to determine the statistical properties of this non-thermal-equilibrium quantum system in the stationary limit of $t \to \infty$. To answer this, the complex P-representation is utilised to obtain a generalised Fokker–Planck equation:

$$\frac{\partial}{\partial t}P(\boldsymbol{\alpha}) = \left\{\frac{\partial}{\partial \alpha}(\kappa\alpha + 2\chi\alpha^2\alpha^+ - \mathscr{E}) - \frac{\partial^2}{\partial \alpha^2} \cdot \alpha^2\chi \right.$$

$$\left. + \frac{\partial}{\partial \alpha^+}(\kappa^*\alpha^+ + 2\chi^*\alpha^{+2}\alpha - \mathscr{E}^*) - \frac{\partial^2}{\partial \alpha^{+2}}\alpha^{+2}\chi^*\right\}P(\boldsymbol{\alpha}) \tag{A3}$$

where

$$\kappa \equiv \kappa' + \mathrm{i}\Delta\omega$$

$$\chi \equiv \chi' + \mathrm{i}\chi''.$$

This has a potential-type solution similar to equation (4.4), even though detuning and nonlinear polarisability are included. For simplicity, the (arbitrary) phase of $\mathscr{E}$ is redefined so that $\mathscr{E}/\chi$ is real and positive. A variable change is then made to variables $\zeta = 1/\alpha$, $\zeta^+ = 1/\alpha^+$. In terms of the new variables, the integration paths are Hankel paths; from $-\infty$ in each variable, around the origin in an anticlockwise direction and then back to $-\infty$. The expectation value of an arbitrary normal-ordered moment of the field operators is then given by:

$$\langle(\hat{a}^+)^m(\hat{a})^n\rangle \propto \iint_C \zeta^{(-n-\mu)}\zeta^{+(-m-\mu^*)} \exp[\mathscr{E}(\zeta+\zeta^+)/\chi + 2/\zeta\zeta^+]\,\mathrm{d}\zeta\,\mathrm{d}\zeta^+$$

$$\propto \iint_C \sum_{j=0}^{\infty} [2^j/j!]\zeta^{(-j-n-\mu)}\zeta^{+(-j-m-\mu^*)} \exp[\mathscr{E}(\zeta+\zeta^+)/\chi]\,\mathrm{d}\zeta\,\mathrm{d}\zeta^+ \tag{A4}$$

where

$$\mu \equiv (\kappa/\chi).$$

These integrals correspond to the definition of the gamma function. Also the infinite series is the defining series for the generalised Gauss hypergeometric function ${}_0F_2$. Including the normalising factor, the final result is

$$\langle(\hat{a}^+)^m(\hat{a})^n\rangle = \frac{|\mathscr{E}/\chi|^{m+n}\Gamma(\mu)\Gamma(\mu^*)\,{}_0F_2(m+\mu^*, n+\mu, |2\mathscr{E}/\chi|)}{\Gamma(m+\mu^*)\Gamma(n+\mu)\,{}_0F_2(\mu^*, \mu, |2\mathscr{E}/\chi|)}. \tag{A5}$$

The above results are obtained using the complex P-representation. Numerical evaluation of the generalised hypergeometric function is straightforward, and results are given in Drummond (1979), Drummond and Walls (1980). However, one simple result that obtains in the low-driving-field limit is the value of $g^2(0)$, which equals $|\kappa/(\kappa+\chi)|^2$. For the absorptive case this displays photon-antibunching, a nonclassical feature of the electromagnetic field. Finally, further results can be calculated using stochastic methods, in the positive P-representation. These are discussed by Chaturvedi *et al* (1977) and Drummond and Walls (1980).

While the stability properties of the equations of motion in this case will be identical to those of the 'naive' stochastic equations, some care is required in the calculations and in simulation of these equations.

References

Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions* (New York: Dover)
Agarwal G S and Wolf E 1970 *Phys. Rev.* D**2** 2161, 2187, 2206
Arnold L 1974 *Stochastic Differential Equations* (New York: Wiley)
Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857, 1882
Carmichael H J and Walls D F 1976 *J. Phys. B: Atom. Molec. Phys.* **9** 1199
Chaturvedi S, Drummond P D and Walls D F 1977 *J. Phys. A: Math. Gen.* **10** L187
Chaturvedi S and Gardiner C W 1978 *J. Statist. Phys.* **18** 501
Drummond P D 1979 *D.Phil Thesis* (University of Waikato)
Drummond P D and Carmichael H J 1978 *Opt. Commun.* **27** 160
Drummond P D, McNeil K J and Walls D F 1979 *Opt. Commun.* **28** 255
Drummond P D and Walls D F 1980 *J. Phys. A: Math. Gen.* **13** 725
Gardiner C W and Chaturvedi S 1977 *J. Statist. Phys.* **17** 429
Glauber R J 1963a *Phys. Rev.* **130** 2529
—— 1963b *Phys. Rev.* **131** 2766
—— 1970 in *Quantum Optics* ed S M Kaye and A Maitland (New York: Academic Press)
Graham R 1977 *Z. Phys.* B**26** 281
Haken H 1970 *Handbuch der Physik* Vol XXV/2c (Berlin: Springer)
Kimble H J, Dagenais M and Mandel L 1978 *Phys. Rev.* A**18** 201
Klauder J R and Sudarshan E C G 1970 *Fundamentals of Quantum Optics* (New York: Benjamin)
Lax M 1968 *Phys. Rev.* **172** 350
Leuchs G, Ratecke M and Walther H 1979 (private communication)
Lonke A 1978 *J. Math. Phys.* **19** 1110
Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)
Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 277